School of Mathematics and Physics
University of Lincoln

Mathematics Challenge–2014

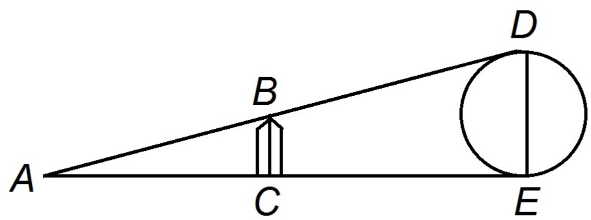
Solutions

Note that each problem may have several different solutions by different methods.

Problem 1. Estimate the distance from which the tower of the Lincoln Cathedral appears of the same size as the diameter of the Sun. Assume that the height of the tower is 83 m.

Solution of Problem 1. We assume that two objects appear for us to be of the same size if the ratios of their size to the distance are the same. This can also be easily seen from the similar triangles in the picture:

(Not to scale!)



An approximate distance to the Sun is $AE \approx 149,600,000\,\text{km}$ and the Sun's diameter is $DE \approx 1,391,600\,\text{km}$. By hypothesis, the height of the tower is $BC = 83\,\text{m} = 0.083\,\text{km}$. By similar triangles we have the proportion

$$\frac{0.083}{AC} = \frac{1,391,600}{149,600,000}.$$

Hence, the sought-for distance is $AC \approx 8.923\,\text{km}$.

Comments on submissions and solutions of Problem 1. When evaluating the submitted solutions, 8.9 km was also accepted as a good enough approximation. There is obviously no need to calculate the angle by using arcsine and then taking sine again, but solutions along this route were also accepted as correct. The visible disc of the Sun has a slightly bigger angular size than the Sun's diameter, but this difference is negligible.

Problem 2. Find the right-most digit of the number 7^{2014} (the 2014-th power of 7).

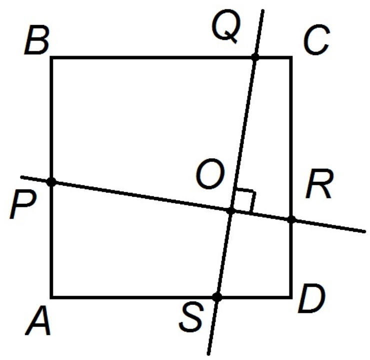
Solution of Problem 2. Calculations apparently easily establish the pattern: $7^1 = ...7$; $7^2 = ...9$; $7^3 = ...3$; $7^4 = ...1$; $7^5 = ...7$; and so on. The important thing to note is that the right-most digit depends only on the right-most digits of the factors, so there is no need to compute the whole power, just multiply the last digits. Clearly, these repeat with period 4. Since $2014 = 503 \cdot 4 + 2$, the last digit of 7^{2014} is the second one in the period, that is, 9.

Problem 3. Find the right-most digit of the number $7^{(7^{2015})}$
(which is 7 to the power of 2015-th power of 7).

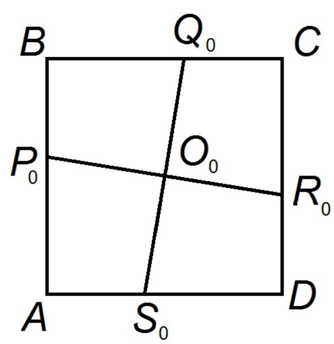
Solution of Problem 3. It follows from the solution of Problem 2 that the last digit of $7^{(7^{2015})}$ is determined by the remainder of 7^{2015} after division by 4. In turn, these remainders actually repeat with period 2. Again, the important thing to note is that the remainder modulo 4 depends only on the remainders modulo 4 of the factors, so there is no need to compute the whole power, just multiply the remainders (and find the remainder of the product of the remainders). Indeed, $(4a + b)(4c + d) = 16ac + 4ad + 4bc + bd = 4(4ac + ad + bc) + bd$, where all numbers are integers. Thus, 7^1 has remainder 3; 7^2 has remainder 1 since $3 \cdot 3 = 9$ has remainder 1; 7^3 has remainder $3 = 1 \cdot 3$; and so on. Clearly, the remainders are repeating with period two as 3, 1, 3, 1, ... Since 2015 is odd, the remainder of 7^{2015} after division by 4 is 3. Then by the solution of Problem 2, the right-most digit of $7^{(7^{2015})}$ is the third in that period 7, 9, 3, 1 of length four, that is, 3.

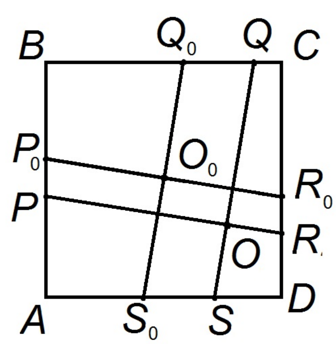
Comments on submissions and solutions of Problem 3. The sought-for remainder of 7^{2015} modulo 4 can also be found by finding the two right-most digits of this number. This is because $100k + m$ has the same remainder modulo 4 as m, since 100 is divisible by 4. The two right-most digits of a product clearly also depend only on the two right-most digits of the factors. This is a bit longer route than described above. Fortunately, in this particular case the two right-most digits repeat with a small period: 7^1 is ...07, 7^2 is ...49, 7^3 is ...43, 7^4 is ...01, and then obviously all will be repeated, clearly, with period 4. Since $2015 = 4k + 3$, the two right-most digits of 7^{2015} are 43, so the remainder of 7^{2015} modulo 4 is 3, as it should be. Then finish as before. Such solutions were also accepted as correct.

Problem 4. Given a square $ABCD$ and a point O inside, there are two perpendicular lines through O. They intersect sides AB in P, BC in Q, CD in R, and DA in S. Thus, four quadrangles are formed: $APOS$, $BQOP$, $CROQ$, and $DSOR$. Prove that the sum of the perimeters of $APOS$ and $CROQ$ is equal to the sum of the perimeters of $BQOP$ and $DSOR$.



Solution of Problem 4. Here is one of possible solutions. In the perimeters of quadrangles under consideration the segments on the lines themselves (like PO, QO, ...) occur in both pairs. Therefore it is only needed to show that $AS + AP + CQ + CR = BP + BQ + DS + DR$. Since the sum of all these segments is always equal to the perimeter of the square, our task is equivalent to showing that $AS + AP + CQ + CR$ is always equal to the half of the perimeter of the square. First consider the special case where the point O is in the centre of the square. The corresponding points are denoted with subscripts 0





as O_0, P_0, ... In this symmetric case, clearly, all four quadrangles are equal and the result is obvious: $AS_0 + AP_0 + CQ_0 + CR_0$ is equal to the half of the perimeter of the square.

Any other general position of the lines can be obtained from such a special position by parallel translation from point O_0 to the new point O. The proof will be complete if we show that the corresponding sum $AS + AP + CQ + CR$ does not change. Indeed, this sum is greater than for O_0 by $S_0S + R_0R$ and smaller by $P_0P + Q_0Q$. But S_0SQQ_0 and P_0PRR_0 are parallelograms by construction and therefore $S_0S = Q_0Q$ and $R_0R = P_0P$. Thus, the sum under consideration increased and decreased by the same amount, that is, did not change, as required.

Comments on submissions and solutions of Problem 4. There are various other ways of proving this fact. For example, introduce the angles between one of the lines and a side of the square, then the same angles will be formed by the other line with other sides of the square. Express the segments AS, etc. by using the sine or cosine of those angles and ultimately all the required sums of the lengths will come out to be equal...

Problem 5. How many sequences of length 10 can be composed of two letters A and B (in various proportions) such that no two letters B stand next to each other?

(For example, $ABAABAAAAB$ is allowed but $ABBAAAAAAA$ is not. You may use binomial coefficients to express your answer.)

Solution of Problem 5. Here is one of the possible solutions. We count separately the number of such words for each proportion of letters A and B. Say, we have k letters A and $10-k$ letters B. Since letters B cannot stand next to one another, any word is determined by the choice of $10-k$ places each for one letter B among $k+1$ possibilities between (and beyond) k letters A (which here are regarded as fixed 'dividers'). For example, for $k=7$ we have

$$\_\, A \,\_\, A \,\_\, A \,\_\, A \,\_\, A \,\_\, A \,\_\, A \,\_$$

and we need to choose 3 places for the letters B from the 8 spaces. The number of such choices is of course a binomial coefficient $\binom{8}{3} = \frac{8!}{3!\,5!} = 56$. (Note that at schools different notation ${}^8C_3$ is often used for the same number.) Thus,

for 0 Bs we have 1 word;

for 1 Bs we have $\binom{10}{1} = 10$ words;

for 2 Bs we have $\binom{9}{2} = 36$ words;

for 3 Bs we have $\binom{8}{3} = 56$ words;

for 4 Bs we have $\binom{7}{4} = 35$ words;

for 5 Bs we have $\binom{6}{5} = 6$ words;

for 6 or more Bs we no longer have any admissible words, since there are only 5 or less places for Bs.

Altogether there are $1 + 10 + 36 + 56 + 35 + 6 = 144$ admissible words.

Comments on submissions and solutions of Problem 5. If the word of all As was excluded, thus giving answer 143, this was accepted as correct, too.

There are some other ways to calculate the number of admissible words. One of the solutions actually submitted is using induction and the recurrence relation (similar to the Fibonacci numbers) for the number $f(i)$ of admissible words of length $i = 1, 2, 3, 4, \ldots$. One can make a certain guess by computing several first values $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 8$, ... More importantly, one can prove that $f(k+2) = f(k) + f(k+1)$. Indeed, every admissible word of length $k+2$ with right-most letter A is obtained from an admissible word of length $k+1$ by adding A on the right, so there are exactly $f(k+1)$ such words. Every admissible word of length $k+2$ with right-most letter B is obtained from an admissible word of length k by adding AB on the right, so there are exactly $f(k)$ such words. Thus, $f(k+2) = f(k) + f(k+1)$. After that it is not difficult to calculate $f(10) = 144$:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.