December 2016 cumulative update for 18th issue of Kourovka Notebook

All the updates are incorporated in the PDF file of the new version of Kourovka Notebook available at `http://arxiv.org/abs/1401.0300` (the Russian version at `http://math.nsc.ru/~alglog/18kt.pdf`).

For convenience of the readers all the updates since the first appearance of the 18th edition are also listed below; the newest ones added in December 2016 are marked **NEW**.

***3.50.** Let G be a group of order $p^\alpha \cdot m$, where p is a prime, p and m are coprime, and let k be an algebraically closed field of characteristic p. Is it true that if the indecomposable projective module corresponding to the 1-representation of G has k-dimension p^α, then G has a Hall p'-subgroup? The converse is trivially true.

A. I. Saksonov

*No, not always: see Example 4.5 in (W. Willems, *Math. Z.*, **171** (1980), 163–174).

7.57. a) *Corrected question*: A set of generators of a finitely generated group G consisting of the least possible number $d(G)$ of elements is called a *basis of* G. Let $r_M(G)$ be the least number of relations necessary to define G in the basis M and let $r(G)$ be the minimum of the numbers $r_M(G)$ over all bases M of G. It is known that $r_M(G) \leqslant d(G) + r(G)$ for any basis M. Does there exist a finitely presented group G for which the inequality becomes equality for some basis M?

V. A. Churkin

***8.10.** a) Is the group $G = \langle a, b \mid a^n = 1,\ ab = b^3a^3 \rangle$ finite or infinite for $n = 7$? All other cases known. See Archive, 7.7 and 8.10 b.

D. L. Johnson

*This group is infinite, because it contains the Fibonacci group $F(3,7)$ as an index 7 subgroup. This follows from Theorem 3.0 of (C. P. Chalk, *Commun. Algebra* **26**, no. 5 (1998), 1511–1546) by standard technique for working with Fibonacci groups (G. Williams, *Letter of 6 October 2015*).

***9.8.** Does there exist a finitely generated simple group of intermediate growth?

R. I. Grigorchuk

*Yes, it does (V. Nekrashevych, `http://arxiv.org/pdf/1601.01033`).

9.75. Find all local formations $\mathfrak{F}$ of finite groups such that, in every finite group, the set of $\mathfrak{F}$-subnormal subgroups forms a lattice.

Comment of 2015: Subgroup-closed formations with this property are described in (Xiaolan Yi, S. F. Kamornikov, *J. Algebra*, **444** (2015), 143–151).

L. A. Shemetkov

***10.23.** Is it true that extraction of roots in braid groups is unique up to conjugation?

G. S. Makanin

*Yes, it is true (J. González-Meneses, *Algebr. Geom. Topology*, **3** (2003), 1103–1118).

***11.11.** a) The well-known Baer–Suzuki theorem states that if every two conjugates of an element a of a finite group G generate a finite p-subgroup, then a is contained in a normal p-subgroup. Does such a theorem hold in the class of periodic groups?

The case $p = 2$ is of particular interest.

A. V. Borovik

*a) No, it does not hold for $p = 2$ (V. D. Mazurov, A. Yu. Ol'shanskiĭ, A. I. Sozutov, *Algebra and Logic*, **54**, no. 2 (2015), 161–166).

11.36(b) The last reference updated: (V. S. Atabekyan, *Math. Notes*, **95**, no. 5 (2014), 586–589).

***11.52.** (Well-known problem). A permutation group on a set Ω is called *sharply doubly transitive* if for any two pairs (α, β) and (γ, δ) of elements of Ω such that $\alpha \neq \beta$ and $\gamma \neq \delta$, there is exactly one element of the group taking α to γ and β to δ. Does every sharply doubly transitive group possess a non-trivial abelian normal subgroup? A positive answer is well known for finite groups. *V. D. Mazurov*

*No, not every (E. Rips, Y. Segev, K. Tent, `arXiv:1406.0382v3`; K. Tent, M. Ziegler, *Adv. Geom.*, **16**, no. 1 (2016), 131–134).

11.70. Reference in the solution of part b) updated: (S. A. Zyubin, *Siberian Electron. Math. Rep.*, **11** (2014), 64–69 (Russian)).

***12.77.** (Well-known problem). Does the order (if it is greater than p^2) of a finite non-cyclic p-group divide the order of its automorphism group? *A. I. Starostin*

*No, not always (J. González-Sánchez, A. Jaikin-Zapirain, *Forum Math. Sigma*, **3**, Article ID e7, 11 p., electronic only (2015)).

***13.45** Every infinite group G of regular cardinality $\mathfrak{m}$ can be partitioned into two subsets $G = A_1 \cup A_2$ so that $A_1F \neq G$ and $A_2F \neq G$ for every subset $F \subset G$ of cardinality less than $\mathfrak{m}$. Is this statement true for groups of singular cardinality?

I. V. Protasov

*No, not always. The answer depends on the algebraic structure of G. In particular, this is true for a free group, but the statement does not hold for every Abelian group G of singular cardinality (I. Protasov, S. Slobodianiuk, *Quest. Answers Gen. Topology*, **33**, no. 2 (2015), 61–70).

14.99. A formation $\mathfrak{F}$ of finite groups is called *superradical* if it is S_n-closed and contains every finite group of the form $G = AB$ where A and B are $\mathfrak{F}$-subnormal F-subgroups.

a) Find all superradical local formations.

NEW *b) Prove that every S-closed superradical formation is a solubly saturated formation. *L. A. Shemetkov*

*b) A counterexample is constructed (S. Yi, S. F. Kamornikov, *Sibirsk. Math. J.*, **57**, no. 2 (2016), 260–264).

NEW ***15.17.** An infinite group is *just infinite* if all of its proper quotients are finite. Is every finitely generated just infinite group of intermediate growth necessarily a branch group? *L. Bartholdi, R. I. Grigorchuk, Z. Šunić*

*No, not every; there exist simple finitely generated groups of intermediate growth (V. Nekrashevych, `http://arxiv.org/pdf/1601.01033`).

15.32. *Rephrased*: Does there exist a function $f(k)$ (possibly depending also on p) such that if a finite p-group G of order p^m with $m \geqslant f(k)$ has an automorphism of order p^{m-k}, then G possesses a cyclic subgroup of index p^k? *Ya. G. Berkovich*

***15.35.** Let F be the free group of finite rank r with basis $\{x_1, \ldots, x_r\}$. Is it true that there exists a number $C = C(r)$ such that any reduced word of length $n > 1$ in the x_i lies outside some subgroup of F of index at most $C \log n$? *O. V. Bogopol'skiĭ*

*No, it is not true (K. Bou-Rabee, D. B. McReynolds, *Bull. London Math. Soc.*, **43**, no. 6 (2011), 1059–1068).

***16.13.** Does there exist a finite p-group G all of whose maximal subgroups H are special, that is, satisfy $Z(H) = [H, H] = \Phi(H)$?

Ya. G. Berkovich

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*Yes, for all primes there are groups of arbitrarily large size with this property (J. Cossey, *Bull. Austral. Math. Soc.*, **89** (2014), 415–419); an example for $p = 2$ given by a Sylow 2-subgroup of $L_3(4)$ was independently presented by V. I. Zenkov at Mal'cev Meeting–2014, 10–14 November, 2014, Novosibirsk.

***16.24.** The *spectrum* of a finite group is the set of orders of its elements. Does there exist a finite group G whose spectrum coincides with the spectrum of a finite simple exceptional group L of Lie type, but G is not isomorphic to L?

A. V. Vasil'ev

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*Yes, for example, for $L = {}^3D_4(2)$ (V. D. Mazurov, *Algebra Logic*, **52**, no. 5 (2013), 400–403). *Reference updated*: Further examples are given in M. A. Grechkoseeva,
NEW M. A. Zvezdina, *J. Algebra Appl.*, **15**, no. 9 (2016), Article ID 1650168, 13 pp.).

16.29. Which finite simple groups of Lie type G have the following property: for every semisimple abelian subgroup A and proper subgroup H of G there exists $x \in G$ such that $A^x \cap H = 1$?

Correction in this paragraph: This property holds when A is a cyclic subgroup (J. Siemons, A. Zalesskii, *J. Algebra*, **256** (2002), 611–625), as well as when A is contained in some maximal torus and $G = PSL_n(q)$ (J. Siemons, A. Zalesskii, *J. Algebra*, **226** (2000), 451–478). Note that if $G = L_2(5)$, $A = 2 \times 2$, and $H = 5 : 2$ (in Atlas notation), then $A^x \cap H > 1$ for every $x \in G$ (this example was communicated to the author by V. I. Zenkov).

E. P. Vdovin

***16.54.** We say that a group G acts *freely* on a group V if $vg \neq v$ for any nontrivial elements $g \in G$, $v \in V$. Is it true that a group G that can act freely on a non-trivial abelian group is embeddable in the multiplicative group of some skew-field?

V. D. Mazurov

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*No, it is not. For example, the group $2.A_5.2$ with a quaternion Sylow 2-subgroup can act freely on an elementary abelian group of order 7^4 but is not embeddable in the multiplicative group of any skew-field by Theorem 7 in (S. A. Amitsur, *Trans. Amer. Math. Soc.*, **80**, no. 2 (1955), 361–386). (D. Nedrenko, *Letter of 20 January 2014*).

NEW ***16.95.** *Conjecture:* If F is a field and A is in $GL(n, F)$, then there is a permutation matrix P such that AP is cyclic, that is, the minimal polynomial of AP is also its characteristic polynomial.

J. G. Thompson

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*The conjecture is proved (J. D. Dixon, Recognizing cyclic matrices and a conjecture of J. G. Thompson, `arxiv.org/abs/1606.02238`).

NEW ***17.17.** If a finitely generated group G has $n < \infty$ maximal subgroups, must G be finite? In particular, what if $n = 3$?

G. M. Bergman

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*No, it must not. An example of an infinite group with 3 maximal subgroups is given by a 2-generated 2-LERF group in § 7 of (M. Ershov, A. Jaikin-Zapirain, *J. Reine Angew. Math.* **677** (2013), 71–134) as its maximal subgroups have index 2.

***17.44.** Let π be a set of primes. A finite group is said to be a C_π-group if it possesses exactly one class of conjugate Hall π-subgroups.

a) In a C_π-group, is an overgroup of a Hall π-subgroup always a C_π-group?

b) In a D_π-group, is an overgroup of a Hall π-subgroup always a D_π-group?

An affirmative answer to (a) in the case $2 \notin \pi$ follows mod CFSG from (F. Gross, *Bull. London Math. Soc.*, **19**, no. 4 (1987), 311–319).

E. P. Vdovin, D. O. Revin

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*a) Yes, it is (E. P. Vdovin, D. O. Revin, *Siberian Math. J.*, **54**, no. 1 (2013), 22–28).

*b) Yes, it is (N. Ch. Manzaeva, `arXiv:1504.03137`).

17.45. A subgroup H of a group G is called *pronormal* if H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$. We say that H is *strongly pronormal* if L^g is conjugate to a subgroup of H in $\langle H, L^g\rangle$ for every $L \leqslant H$ and $g \in G$.

*(a) In a finite simple group, are Hall subgroups always pronormal?

(b) In a finite simple group, are Hall subgroups always strongly pronormal?

Comment of 2015: a counterexample to (b) is announced in (M. N. Nesterov, *Abstracts of Mal'cev Meeting–2015*, Novosibirsk, 2015, p. 114 (electronic, Russian); `http://www.math.nsc.ru/conference/malmeet/15/malmeet15.pdf`).

(c) In a finite group, is a Hall subgroup with a Sylow tower always strongly pronormal?

An affirmative answer to (a) is known modulo CFSG for Hall subgroups of odd order (F. Gross, *Bull. London Math. Soc.*, **19**, no. 4 (1987), 311–319). Notice that there exist finite (non-simple) groups with a non-pronormal Hall subgroup. Hall subgroups with a Sylow tower are known to be pronormal.

E. P. Vdovin, D. O. Revin

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*(a) Yes, they are (E. P. Vdovin, D. O. Revin, *Siberian Math. J.*, **53**, no 3 (2012), 419–430).

17.46. Reference in the solution updated: (B. Wilkens, *J. Group Theory*, **17**, no. 1 (2014), 151–174).

17.67. (H. Zassenhaus). *Conjecture*: Every invertible element <u>of finite order</u> of the integral group ring $\mathbb{Z}G$ of a finite group G is conjugate in the rational group ring $\mathbb{Q}G$ to an element of $\pm G$.

V. D. Mazurov

17.86. Reference in the solution of part b) updated: (V. A. Roman'kov, *Siberian Math. J.*, **52**, no. 2 (2011), 348–351).

***18.9.** Does there exist a subgroup-closed saturated formation $\mathfrak{F}$ of finite groups properly contained in $\mathfrak{E}_\pi$, where $\pi = char(\mathcal{F})$, satisfying the following property: if $G \in \mathfrak{F}$, then there exists a prime p (depending on the group G) such that the wreath product $C_p \wr G$ belongs to $\mathfrak{F}$, where C_p is the cyclic group of order p?

A. Ballester-Bolinches

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*Yes, it exists. Let $\pi = \{2, 3, 7\}$ and let $\mathfrak{F}$ be the class of all groups that are extensions of soluble π-groups by finite direct products of groups isomorphic to $PSL_2(7)$. Then $\mathfrak{F}$ is a saturated formation equal to $\mathfrak{F} = \mathfrak{S}_\pi form(PSL_2(7))$ and $char(\mathfrak{F}) = \pi$. Since all proper subgroups of $PSL_2(7)$ are soluble, it follows that $\mathfrak{F}$ is a hereditary formation. Since the sets of prime divisors of the orders of the groups $PSL_2(2^3)$ and $PSU_3(3)$ coincide with π, it follows that $\mathfrak{F}$ is a proper subformation of $\mathfrak{E}_\pi$. Furthermore, if $G \in \mathfrak{F}$, then the wreath product $C_p \wr G$ belongs to $\mathfrak{F}$, where C_p is a cyclic group of order $p \in \pi$. (S. F. Kamornikov, *Letter of 20 January 2014*).

18.32. Is every Hall subgroup of a finite group pronormal in its normal closure?

Comment of 2015: a counterexample is announced in (M. N. Nesterov, *Abstracts of Mal'cev Meeting–2015*, Novosibirsk, 2015, p. 114 (electronic, Russian); `http://www.math.nsc.ru/conference/malmeet/15/malmeet15.pdf`).

E. P. Vdovin, D. O. Revin

***18.33.** A group in which the derived subgroup of every 2-generated subgroup is cyclic is called an *Alperin group.* Is there a bound for the derived length of finite Alperin groups?

G. Higman proved that finite Alperin groups are soluble, and finite Alperin p-groups have bounded derived length, see 17.46.

B. M. Veretennikov

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*Yes, there is: it at at most 6. Indeed, in (P. Longobardi, M. Maj, H. Smith, *Rend. Semin. Mat. Univ. Padova*, **115** (2006), 29–40) it was proved that finite Alperin groups of odd order are metabelian, and any finite Alperin group is supersoluble, so the elements of odd order form a metabelian normal subgroup, while finite Alperin 2-groups have derived length at most 4 by (B. Wilkens, *J. Group Theory*, **17**, no. 1 (2014), 151–174).

***18.49.** Let $n \in \mathbb{N}$. Is it true that for any $a, b, c \in \mathbb{N}$ satisfying $1 < a, b, c \leqslant n-2$ the symmetric group S_n has elements of order a and b whose product has order c?

S. Kohl

NEW *Yes, it is (J. König, *Eur. J. Comb.*, **57** (2016), 50–56). (*Reference updated.)*

***18.52.** Is every finite simple group generated by two elements of prime-power orders m, n? (Here numbers m, n may depend on the group.) The work of many authors shows that it remains to verify this property for a small number of finite simple groups.

J. Krempa

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*Yes, it is (mod CFSG); moreover, every finite simple group is generated by an involution and an element of prime order (C. S. H. King, `arXiv:1603.04717v1`).

NEW **18.55.** (a) *Correction: "of finite exponent" added*: Can a locally finite p-group G of finite exponent be the union of conjugates of an abelian proper subgroup?

(b) Can this happen when G is of exponent p?

G. Cutolo

18.56. Let G be a finite 2-group, of order greater than 2, such that $|H/H_G| \leq 2$ for all $H \leq G$, where H_G denotes the largest normal subgroup of G contained in H. Must G have an abelian subgroup of index 4?

G. Cutolo

***18.57.** Let G be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of G bounded?

D. V. Lytkina

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*No, it is not (A. Abdollahi, *J. Group Theory*, **18**, no. 1 (2015), 111–114).

18.65. *Corrected version*: (R. Guralnick, G. Malle). *Conjecture*: Let p be a prime different from 5, and C a class of conjugate p-elements in a finite group G. If $[c, d]$ is a p-element for any $c, d \in C$, then $C \subseteq O_p(G)$.

V. D. Mazurov

***18.75.** Does every finite solvable group G have the following property: there is a number $d = d(G)$ such that G is a homomorphic image of every group with d generators and one relation?

This property holds for finite nilpotent groups and does not hold for every non-solvable finite group; see (S. A. Zaĭtsev, *Moscow Univ. Math. Bull.*, **52**, no. 4 (1997), 42–44).

A. Yu. Ol'shanskiĭ

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*Yes, it does (N. Nikolov, D. Segal, *Bull. London Math. Soc.*, **39**, no. 2 (2007), 209–213).

18.81. *Rephrased*: Let G be a finitely generated p-group that is residually finite. Are all maximal subgroups of G necessarily normal?

D. S. Passman

***18.82.** Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that for any prime p, if $p^{f(n)}$ divides the order of a finite group G, then p^n divides the order of $\operatorname{Aut} G$?

R. M. Patne

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*Yes, there is. This was established in (J. A. Green, *Proc. Roy. Soc. London Ser. A.*, **237** (1956), 574–581), which improved a previous result with a function depending on p in (W. Ledermann, B. H. Neumann, *Proc. Roy. Soc. London. Ser. A*, **235** (1956), 235–246).

***18.86.** Is the group $G = \langle a, b \mid [[a, b], b] = 1\rangle$, which is isomorphic to the group of all unitriangular automorphisms of the free group of rank 3, linear?

V. A. Roman'kov

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*Yes, it is linear, since it embeds in the holomorph $\operatorname{Hol} F_2$ of the free group F_2, which can be seen by adding a new generator $c = [a, b]$, so that $G = \langle a, b, c \mid a^b = ac,\ c^b = c\rangle$, while $\operatorname{Hol} F_2$ was shown to be linear in Corollary 3 in (V. G. Bardakov, O. V. Bryukhanov, *Vestnik Novosibirsk Univ. Ser. Mat. Mekh. Inf.*, **7**, no. 3 (2007), 45–58 (Russian)) (O. V. Bryukhanov, *Letter of 27 January 2014*). Another proof can be found in (V. A. Roman'kov, *J. Siberian Federal Univ. Math. Phys.*, **6**, no. 4 (2013), 516–520).

***18.91.** A subgroup H of a group G is said to be *propermutable* in G if there is a subgroup $B \leqslant G$ such that $G = N_G(H)B$ and H permutes with every subgroup of B.

(a) Is there a finite group G with subgroups $A \leqslant B \leqslant G$ such that A is propermutable in G but A is not propermutable in B?

(b) Let P be a non-abelian Sylow 2-subgroup of a finite group G with $|P| = 2^n$. Suppose that there is an integer k such that $1 < k < n$ and every subgroup of P of order 2^k is propermutable in G, and also, in the case of $k = 1$, every cyclic subgroup of P of order 4 is propermutable in G. Is it true that then G is 2-nilpotent?

A. N. Skiba

NEW *(a) Yes, there is (A. A. Pypka, D. Yu. Storozhenko, On some types of propermutable subgroups and their generalizations in finite groups, *to appear in Dopov. Nac. Akad. Nauk Ukrain.*).

*(b) Yes, it is (Kh. A. Al-Sharo, Finite groups with given systems of weakly S-propermutable subgroups, *J. Group Theory*, **19**, no. 5, 871–887).

NEW ***18.94.** Let G be a group without involutions, a an element of it that is not a square of any element of G, and n an odd positive integer. Is it true that the quotient $G/\langle (a^n)^G\rangle$ does not contain involutions?

A. I. Sozutov

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*No, not always, as shown by an example of V. I. Trofimov; another example: the quotient of $G = \langle a, b \mid a^2 = b^2\rangle$ by $\langle a^G\rangle$ has order 2.

NEW ***18.95.** Suppose that a group $G = AB$ is a product of an abelian subgroup A and a locally quaternion group B (that is, B is a union of an increasing chain of finite generalized quaternion groups). Is G soluble?

A. I. Sozutov

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*Yes, it is (B. Amberg, Ya. Sysak, On products of groups with abelian subgroups of small index, `https://arxiv.org/abs/1611.10093`).

18.101. *Editors' apology*: this is the same as 17.11.

18.113. Corrected version: Let $\mathfrak{M}$ be a finite set of finite simple nonabelian groups. Is it true that a periodic group saturated with groups from $\mathfrak{M}$ (see 14.101) is isomorphic to one of groups in $\mathfrak{M}$? The case where $\mathfrak{M}$ is one-element is of special interest.

A. K. Shlëpkin