Almost Engel finite, profinite, and compact groups

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Engel groups

Notation: left-normed simple commutators

$$[a_1, a_2, a_3, \ldots, a_r] = [\ldots[[a_1, a_2], a_3], \ldots, a_r].$$

Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x, g, g, \ldots, g] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

Known facts on finite groups

Zorn's Theorem

A finite Engel group is nilpotent.

Proof:

Coprime action $\Rightarrow$ non-Engel.

No coprime action $\Rightarrow$ nilpotent. $\square$

Baer's Theorem

If g is an Engel element of a finite group G,
that is, $[x, g, \ldots, g] = 1$ for every $x \in G$, then $g \in F(G)$.

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.

Engel compact groups

J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zel'manov's Theorem

If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Almost Engel groups

Definition

A group G is almost Engel if for every $g \in G$ there is a finite set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \ldots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

Includes Engel groups: when $\mathscr{E}(g) = \{1\}$ for all $g \in G$.

Theorem 1 (Almost Engel $\Rightarrow$ almost locally nilpotent)

Suppose that G is an almost Engel compact (Hausdorff) group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(...And there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)

Three parts of the proof

1. **Finite groups**, a quantitative version.
2. **Profinite groups:** using finite groups, Wilson–Zelmanov theorem.
3. **Compact groups:** reduction to profinite case
   using structure theorems for compact groups.

Some notation

If G is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathscr{E}(g)$ with the property that for every $x \in G$,

$$[x, \underbrace{g, g, \dots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g)$$

(for possibly larger numbers $n(x, g)$).

We fix the symbols $\mathscr{E}(g)$ for these minimal sets, call them Engel sinks.

The nilpotent residual of a group G is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).

Almost Engel finite groups

For finite groups there must be a quantitative analogue of the hypothesis that the sinks $\mathscr{E}(g)$ are finite.

Theorem 2

Suppose that G is a finite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$. Then $|\gamma_\infty(G)|$ is bounded in terms of m.

(...And G also has a nilpotent normal subgroup of bounded index: $C_G(\gamma_\infty(G))$.)

Theorem 2 can be viewed as a generalization of Zorn's theorem that a finite Engel group is nilpotent: almost Engel $\Rightarrow$ almost nilpotent.

About the proof for finite groups

Lemma

In any almost Engel group G, a (minimal) Engel sink is the set

$$\mathscr{E}(g) = \{z \in G \mid z = [z, g, \ldots, g]\}$$

(with at least one occurrence of g).

Indeed, $x \to [x, g]$ is a mapping of $\mathscr{E}(g)$ into itself,

must be "onto" since $\mathscr{E}(g)$ is finite and minimal,

$z \in \mathscr{E}(g)$ belongs to its orbit. □

Lemma

In a finite group, if A is an abelian section, acted on by g of coprime order, then $[A, g] = \{[a, g, \ldots, g] \mid a \in A\}$ for any number of g, so $[A, g] \subseteq \mathscr{E}(g)$.

Proof: $C_{[A,g]}(g) = 1 \quad \Rightarrow \quad [A, g] = \{[b, g] \mid b \in [A, g]\}$. □

About the proof for finite groups

Lemma

Let V be an elementary abelian q-group, and U a q'-group of automorphisms of V. If $|[V,u]| \leqslant m$ for every $u \in U$, then $|[V,U]|$ is m-bounded, and therefore $|U|$ is also m-bounded.

Lemma

If $|\mathscr{E}(g)| \leqslant m$ for all $g \in G$, then $G/F(G)$ is of m-bounded exponent.

Proof: Clearly, g centralizes its powers. Hence for any $z \in \mathscr{E}(g^k)$ we have

$$z = [z, g^k, \dots, g^k] \quad \Rightarrow \quad z^g = [z^g, g^k, \dots, g^k].$$

Therefore $\mathscr{E}(g^k)$ is g-invariant. Choose $k = m!$. Then $g^{m!}$ centralizes $\mathscr{E}(g^{m!})$, hence $\mathscr{E}(g^{m!}) = \{1\}$ in fact, so $g^{m!}$ is an Engel element. By Baer's theorem, then $g^{m!} \in F(G)$, so $G/F(G)$ has exponent dividing $m!$. $\square$

Further proof for finite groups

Proposition

If $\forall\ |\mathscr{E}(g)| \leqslant m$, *then* $|G/F(G)|$ *is m-bounded.*

First for the case of soluble G.

Then considering the generalized Fitting subgroup = socle of $G/S(G)$ (using CFSG).

Finally Theorem 1 (that $|\gamma_\infty(G)|$ is m-bounded)

is proved by induction on $|G/F(G)|$...

Profinite groups

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-p groups.

Largest normal pronilpotent subgroup (closed).

Lemma

A pronilpotent almost Engel group H is in fact an Engel group.

Proof: For any $h \in H$ there is a normal subgroup R such that $\mathscr{E}(h) \cap R = \{1\}$ with nilpotent H/R.

Then $\mathscr{E}(h) \subseteq R$, so in fact $\mathscr{E}(h) = \{1\}$,

so h is an Engel element. $\square$

Bounded version for profinite groups

Theorem 2 on finite groups immediately implies the following.

Corollary

Suppose that G is an almost Engel profinite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$.
Then G has a finite normal subgroup N <u>of order bounded in terms of m</u> such that G/N is locally nilpotent.

Proof: In each finite quotient, γ_∞ has m-bounded order by Theorem 2.

Then $N = \gamma_\infty(G)$ has m-bounded order.

G/N is pronilpotent; by Lemma above is an Engel group.

Then by Wilson–Zelmanov theorem G/N is locally nilpotent. □

General case of profinite groups

Theorem 3

Suppose that G is an almost Engel profinite group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Cannot simply apply Theorem 2 on finite groups – as there is no apriori uniform bound on $|\mathscr{E}(g)|$.

First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.

A piece of proof

Lemma

In an almost Engel profinite group G, the sets

$$E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$$

are closed.

Proof: For $y \notin E_k$ we have $|\mathscr{E}(y)| \geqslant k+1$, so there are $z_1, z_2, \dots, z_{k+1}$ distinct elements, each

$$z_i = [z_i, y, \dots, y]. \tag{*}$$

There is an open normal subgroup N such that the images of the z_i are distinct in the finite quotient G/N.

Then equations (*) show that for every $n \in N$ the sink $\mathscr{E}(yn)$ has an element in every coset $z_i N$, whence $|\mathscr{E}(yn)| \geqslant k+1$. So yN is also contained in $G \setminus E_k$. Thus, $G \setminus E_k$ is open, so E_k is closed. □

Application of Baire theorem

Recall: $E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$ are closed.

In the theorem, G is almost Engel, which means $G = \bigcup E_k$.

By the Baire category theorem, one of E_k contains an open set, coset aU, where U is an open subgroup.

This gives us, in a certain metabelian section, a uniform bound for $|\mathscr{E}(u)|$ for all $u \in U$, and then Theorem 2 on finite groups can be applied...

Thus, $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above). Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.

Compact groups

Recall

Theorem 1

Suppose that G is an almost Engel compact group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Structure theorems for compact groups:

- The connected component G_0 of the identity is a divisible group (that is, for every $g \in G_0$ and every integer k there is $h \in G_0$ such that $h^k = g$).
- $G_0/Z(G_0)$ is a Cartesian product of simple compact Lie groups.
- G/G_0 is a profinite group.

Note that a simple compact Lie group is a linear group.

G_0 is abelian

Lemma

An almost Engel divisible group is an Engel group.

Proof: For $g \in G_0$, let $|\mathscr{E}(g)| = m$. Choose $h \in G_0$ such that $h^{m!} = g$. Clearly, h centralizes g, so for any $z \in \mathscr{E}(g)$ we have

$$z = [z, g, \dots, g] \quad \Rightarrow \quad z^h = [z^h, g, \dots, g].$$

Hence $\mathscr{E}(g)$ is h-invariant. Then $h^{m!} = g$ centralizes $\mathscr{E}(g)$. This means that actually $\mathscr{E}(g) = \{1\}$, so g is an Engel element. $\square$

By the structure theorem, G_0 is divisible, so is Engel by the above. By well-known results (Garashchuk–Suprunenko, 1960) linear Engel groups are locally nilpotent. Hence $Z(G_0) = G_0$ is abelian by the structure theorem.

Using the profinite case

We apply Theorem 3 on profinite groups to G/G_0.

Thus we have $G_0 < F < G$ with G_0 abelian divisible, F/G_0 finite, and G/F locally nilpotent.

Next steps:

$\mathscr{E}(g) \cap G_0 = \{1\}$ for all $g \in G$;

$[G_0, \mathscr{E}(g)] = 1$ for all $g \in G$;

Replace (rename) F by possibly smaller subgroup $\langle \mathscr{E}(g) \mid g \in G \rangle G_0$, so $G_0 \leqslant Z(F)$;

... etc., in the end use Theorem 3 on profinite again.

Rank restriction (work in progress)

Instead of being finite, suppose that $\mathscr{E}(g)$ generates a subgroup of finite (Prüfer) **rank**, for all $g \in G$.

Conjecture:

If G is a compact (or profinite) group, then there is a normal closed subgroup N of finite rank such that G/N is locally nilpotent.

So far, the case of finite groups seems to have been done:

Theorem 4

Suppose that G is a finite group and there is a positive integer r such that $\langle \mathscr{E}(g) \rangle$ has rank at most r for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of r.