Almost Engel compact groups

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Norddeutsches Gruppentheorie-Kolloquium 2018
Braunschweig, 29–30 June 2018

Joint work with Pavel Shumyatsky

Engel groups

Notation: left-normed simple commutators

$$[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r].$$

Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x, g, g, \dots, g] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

Known facts on finite groups

Zorn's Theorem

A finite Engel group is nilpotent.

Proof:

Coprime action $\Rightarrow$ non-Engel.

No coprime action $\Rightarrow$ nilpotent. $\square$

Baer's Theorem

If g is an Engel element of a finite group G,
that is, $[x, g, \dots, g] = 1$ for every $x \in G$, then $g \in F(G)$.

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.

Engel compact groups

J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zelmanov's Theorem

If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Almost Engel groups

Definition

A group G is almost Engel if for every $g \in G$ there is a <u>finite</u> set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \ldots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

Includes Engel groups: when $\mathscr{E}(g) = \{1\}$ for all $g \in G$.

Theorem 1 (almost Engel $\Rightarrow$ almost locally nilpotent)

Suppose that G is an almost Engel compact (Hausdorff) group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(...there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)

Three parts of the proof

1. **Finite groups**, a quantitative version.
2. **Profinite groups:** using finite groups, Wilson–Zelmanov theorem.
3. **Compact groups:** reduction to profinite case
   using structure theorems for compact groups.

Some notation

If G is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathscr{E}(g)$ with the property that for every $x \in G$,

$$[x, \underbrace{g, g, \ldots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g)$$

(for possibly larger numbers $n(x, g)$).

We fix the symbols $\mathscr{E}(g)$ for these minimal sets, call them Engel sinks.

The nilpotent residual of a group G is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).

Almost Engel finite groups

For finite groups there must be a quantitative analogue of the hypothesis that the sinks $\mathscr{E}(g)$ are finite.

Theorem 2

Suppose that G is a finite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$. Then $|\gamma_\infty(G)|$ is bounded in terms of m.

(G also has a nilpotent normal subgroup of bounded index: $C_G(\gamma_\infty(G))$.)

About the proof for finite groups

Lemma

In any almost Engel group G, the Engel sink is the set

$$\mathscr{E}(g) = \{z \in G \mid z = [z, g, \dots, g]\}$$

(with at least one occurrence of g).

Indeed, $x \to [x, g]$ is a mapping of $\mathscr{E}(g)$ into itself,

must be "onto" since $\mathscr{E}(g)$ is finite and minimal,

$z \in \mathscr{E}(g)$ belongs to its orbit. $\square$

Lemma

In a finite group, if A is an abelian section, acted on by g of coprime order, then $[A, g] = \{[a, g, \dots, g] \mid a \in A\}$ for any number of g, so $[A, g] \subseteq \mathscr{E}(g)$.

Proof: $C_{[A,g]}(g) = 1 \quad \Rightarrow \quad [A, g] = \{[b, g] \mid b \in [A, g]\}$. $\square$

About the proof for finite groups

Lemma

If $|\mathscr{E}(g)| \leqslant m$ for all $g \in G$, then $G/F(G)$ is of m-bounded exponent.

Proof: Clearly, g centralizes its powers. Hence for any $z \in \mathscr{E}(g^k)$ we have

$$z = [z, g^k, \dots, g^k] \quad \Rightarrow \quad z^g = [z^g, g^k, \dots, g^k].$$

Therefore $\mathscr{E}(g^k)$ is g-invariant.

Choose $k = m!$. Then $g^{m!}$ centralizes $\mathscr{E}(g^{m!})$,
hence $\mathscr{E}(g^{m!}) = \{1\}$ in fact, so $g^{m!}$ is an Engel element.

By Baer's theorem, then $g^{m!} \in F(G)$, so $G/F(G)$ has exponent dividing $m!$. $\square$

Further proof for finite groups

Proposition

If $\forall\ |\mathscr{E}(g)| \leqslant m$, *then* $|G/F(G)|$ *is m-bounded.*

First for the case of soluble G.

Then considering the generalized Fitting subgroup = socle of $G/S(G)$ (using CFSG)......

Proof of Theorem 2 (that $|\gamma_\infty(G)|$ is m-bounded)

is by induction on $|G/F(G)|$...

Profinite groups

Recall:

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-p groups.

Largest normal pronilpotent subgroup (closed).

Lemma

A pronilpotent almost Engel group H is in fact an Engel group.

Proof: For any $h \in H$ there is a normal subgroup R such that $\mathscr{E}(h) \cap R = \{1\}$ with nilpotent H/R.

Then $\mathscr{E}(h) \subseteq R$, so in fact $\mathscr{E}(h) = \{1\}$,

so h is an Engel element. $\square$

Bounded version for profinite groups

Theorem 2 on finite groups immediately implies the following.

Corollary

Suppose that G is an almost Engel profinite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$.
Then G has a finite normal subgroup N <u>of order bounded in terms of m</u> such that G/N is locally nilpotent.

General case of profinite groups

Theorem 3

Suppose that G is an almost Engel profinite group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Cannot simply apply Theorem 2 on finite groups – as there is no apriori uniform bound on $|\mathscr{E}(g)|$.

First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.

A piece of proof

Lemma

In an almost Engel profinite group G, the sets

$$E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$$

are closed.

Proof: For $y \notin E_k$ we have $|\mathscr{E}(y)| \geqslant k+1$, so there are $z_1, z_2, \ldots, z_{k+1}$ distinct elements, each

$$z_i = [z_i, y, \ldots, y]. \tag{*}$$

There is an open normal subgroup N such that the images of the z_i are distinct in the finite quotient G/N.

Then equations (*) show that for every $n \in N$ the sink $\mathscr{E}(yn)$ has an element in every coset z_iN, whence $|\mathscr{E}(yn)| \geqslant k+1$. So yN is also contained in $G \setminus E_k$. Thus, $G \setminus E_k$ is open, so E_k is closed. $\square$

Application of Baire theorem

Recall: $E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$ are closed.

In the theorem, G is almost Engel, which means $G = \bigcup E_k$.

By the Baire category theorem, one of E_k contains an open set, coset aU, where U is an open subgroup.

This gives us, in a certain metabelian section, a uniform bound for $|\mathscr{E}(u)|$ for all $u \in U$, and then Theorem 2 on finite groups can be applied...

Thus, $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above).

Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.

Compact groups

Recall

Theorem 1

Suppose that G is an almost Engel compact group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Structure theorems for compact groups:

- The connected component G_0 of the identity is a divisible group (that is, for every $g \in G_0$ and every integer k there is $h \in G_0$ such that $h^k = g$).
- $G_0/Z(G_0)$ is a Cartesian product of simple compact Lie groups.
- G/G_0 is a profinite group.

Note that a simple compact Lie group is a linear group.

G_0 is abelian

Lemma

An almost Engel divisible group is an Engel group.

Proof: For $g \in G_0$, let $|\mathscr{E}(g)| = m$. Choose $h \in G_0$ such that $h^{m!} = g$. Clearly, h centralizes g, so for any $z \in \mathscr{E}(g)$ we have

$$z = [z, g, \ldots, g] \quad \Rightarrow \quad z^h = [z^h, g, \ldots, g].$$

Hence $\mathscr{E}(g)$ is h-invariant. Then $h^{m!} = g$ centralizes $\mathscr{E}(g)$. This means that actually $\mathscr{E}(g) = \{1\}$, so g is an Engel element. □

By the structure theorem, G_0 is divisible, so is Engel by the above.

By well-known results (Garashchuk–Suprunenko, 1960),
linear Engel groups are locally nilpotent.

Hence $Z(G_0) = G_0$ is abelian by the structure theorem.

Using the profinite case

We apply Theorem 3 on profinite groups to G/G_0.

Thus we have $G_0 < F < G$ with G_0 abelian divisible, F/G_0 finite, and G/F locally nilpotent.

Next steps:

$\mathscr{E}(g) \cap G_0 = \{1\}$ for all $g \in G$;

$[G_0, \mathscr{E}(g)] = 1$ for all $g \in G$;

Replace (rename) F by possibly smaller subgroup $\langle \mathscr{E}(g) \mid g \in G \rangle G_0$, so $G_0 \leqslant Z(F)$;

... etc., in the end use Theorem 3 on profinite again.

Almost Engel in the sense of rank

Instead of being finite, suppose that $\mathscr{E}(g)$ generates a subgroup of finite (Prüfer) **rank**, for all $g \in G$.

Conjecture:

If G is a compact (or profinite) group, then there is a normal closed subgroup N of finite rank such that G/N is locally nilpotent.

So far, the case of finite groups has been done:

Theorem 4

Suppose that G is a finite group and there is a positive integer r such that $\langle \mathscr{E}(g) \rangle$ has rank at most r for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of r.

Engel-type subgroups in finite groups and some length parameters

To measure 'deviation from being n-Engel':

Definition

$$E_n(g) = \langle [x, \underbrace{g, \dots, g}_{n}] \mid x \in G \rangle.$$

Remark: Note that this is not a subnormal subgroup, unlike the subgroups

$$G \trianglerighteq [G, g] \trianglerighteq [[G, g], g] \trianglerighteq \cdots$$

Soluble groups

Recall: Fitting series: $F_1(G) = F(G)$ largest normal nilpotent, then $F_{k+1}(G)/F_k(G) = F(G/F_k(G))$.

If G is finite soluble, then the least h such that $F_h(G) = G$ is the *Fitting height* of G.

Theorem 5

If g is an element of a soluble finite group G such that $E_n(g)$ (for some n) has Fitting height k, then $g \in F_{k+1}(G)$.

The proof of Theorem 1 reduces to the following proposition.

Proposition

Let α be an automorphism of a finite soluble group G such that $G = [G, \alpha]$. Then $E_n(\alpha) = G$ for any n.

(Here, $E_n(\alpha)$ is a subgroup of $G\langle\alpha\rangle$.)

Generalized Fitting height

The generalized Fitting series of a finite group G starts from the generalized Fitting subgroup $F_1^*(G) = F^*(G)$, which the product of the Fitting subgroup and all quasisimple subnormal subgroups, and by induction $F_{i+1}^*(G)/F_i^*(G) = F^*(G/F_i^*(G))$.

The generalized Fitting height $h = h^*(G)$ of a finite group G is the least h such that $F_h^*(G) = G$.

Theorem 6

If g is an element of a finite group G such that $E_n(g)$ (for some n) has generalized Fitting height k, then $g \in F_{f(k,m)}^*(G)$, where m is the number of prime divisors of $|g|$.

(In fact, $f(k,m) = ((k+1)m(m+1)+2)(k+3)/2$.)

Non-soluble length

The nonsoluble length $\lambda(G)$ of a finite group G is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups.

Similarly to the generalized Fitting series, we can define terms of the 'upper nonsoluble series': $R_i(G)$ is the maximal normal subgroup of G that has nonsoluble length i.

Theorem 7

Let m and n be positive integers, and let g be an element of a finite group G whose order $|g|$ is equal to the product of m primes counting multiplicities. If the nonsoluble length of $E_n(g)$ is equal to k, then g belongs to $R_{g(k,m)}(G)$.

(In fact, $g(k,m) = (k+1)m(m+1)/2$.)

Importance of generalized Fitting height and nonsoluble length

Bounds for the nonsoluble length and/or generalized Fitting height greatly facilitate using the classification (and are themselves often obtained by using the classification).

Examples:

- reduction of the Restricted Burnside Problem to soluble and nilpotent groups in the Hall–Higman paper;
- Wilson's reduction of the problem of local finiteness of periodic profinite groups to pro-p groups;

(Both the Restricted Burnside Problem and the problem of local finiteness of periodic profinite groups were solved by Zelmanov.)

- our recent paper of EKh–Shumyatsky on similar problems about profinite groups.

About the proofs of nonsoluble results

Theorem 5 on generalized Fitting height follows from Theorem 6 on nonsoluble length and Theorem 4 on soluble groups.

The proof of Theorem 6 depends on the classification of finite simple group in so far as the validity of the Schreier conjecture on solubility of the group of outer automorphisms of a finite simple group.

One of the ingredients are properties of automorphisms of direct products of nonabelian finite simple groups. A typical lemma:

Lemma

Let $S = S_1 \times \cdots \times S_r$ be a direct product of r isomorphic finite non-abelian simple groups and let φ be the natural automorphism of S of order r that regularly permutes the S_i. Let n be a positive integer. Then $E_n(\varphi) = S$.

Exact orbits

An important role in the proof is played by results on permutational actions of certain finite groups G producing exact (regular) orbits of an element $g \in G$.

Corresponding lemmas rather too technical to be presented here...

Open problems and conjectures

In the 'nonsoluble' theorems the functions depend on the number of prime divisors of $|g|$. We conjecture that this dependence can be eliminated. Moreover, we have quite precise conjectures (with best-possible bounds):

Conjecture 1

Let g be an element of a finite group G, and n a positive integer. If the generalized Fitting height of $E_n(g)$ is equal to k, then $g \in F^*_{k+1}(G)$.

Conjecture 2

Let g be an element of a finite group G, and n a positive integer. If the nonsoluble length of $E_n(g)$ is equal to k, then $g \in R_k(G)$.

Reduction of conjectures

Question

Let $S = S_1 \times \cdots \times S_r$ be a direct product of nonabelian finite simple groups, and φ an automorphism of S transitively permuting the factors. Is it true that $E_n(\varphi) = S$ for any n?

Thus, our Lemma above gives an affirmative answer in the special case where $|\varphi| = r$.

Theorem 8

Conjectures 1 *and* 2 *are true if the Question has an affirmative answer.*

Some progress was made for the Question in the case where $|\varphi|$ is a prime by Robert Guralnick (unpublished).