Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q - 1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

Some aspects of thin Lie algebras

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Characteristic two
Sandwiches

Summary

1. Background and motivations
    - Graded Lie algebras of finite coclass
    - Graded Lie algebras of maximal class
2. Thin Lie algebras
    - Thinness
    - Second diamond in degree $2q - 1$
    - Second diamond in degree q
3. Some special aspects of thin Lie algebras
    - Thin Lie algebras of characteristic two
    - Sandwiches

Coclass for graded Lie algebras

Definition

In this talk, a *graded Lie algebra* (or *gLa*) is an $\mathbb{N}$-graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$, over a field F, such that L_1 has finite dimension and generates L.

- Note that $L^j = \bigoplus_{i \geq j} L_i$.

Definition

The *coclass* of L is the smallest integer r such that $\dim(L/L^i) \leq i + r$ for all i, if it exists, ∞ otherwise.

- A gLa has finite coclass if and only if $\dim(L_i) \leq 1$ from some point on.
- A gLa L of coclass 1 is called *of maximal class* (*gLamc*).

A coclass theory in characteristic zero

Example (The metabelian gLamc)

$M = \bigoplus_{i\geq 1} M_i$, where $M_1 = Fx \oplus Fy_1$, and $M_i = Fy_i$ for $i > 1$,

$$[y_i, x] = y_{i+1}, \qquad [y_i, y_j] = 0,$$

is the unique infinite-dim gLamc in char zero (Vergne 1966).

Theorem (Shalev-Zelmanov 1997)

There is a function f such that, if L is a gLa of coclass r of char zero, then $L/Z_{f(r)}(L) \cong M/M^d$, for some d.

A. Shalev and E.I. Zelmanov
Narrow Lie algebras: a coclass theory and a characterization of the Witt algebra
J. Algebra **189** (1997), 294–331

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q-1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

Some insoluble gLamc in prime characteristic

- Shalev and Zelmanov's result implies analogues for gLa of char zero of the coclass conjectures for pro-p groups.
- In essence for us: gLa of finite coclass in char zero are soluble.
- However, in char $p > 0$ this fails already in the case of coclass one:

Theorem (Shalev 1994)

For each prime p there exists countably many non-soluble gLamc L over $\mathbb{F}_p$.

- Shalev's algebras do have a periodic structure.

A. Shalev
Simple Lie algebras and Lie algebras of maximal class
Arch. Math. **63** (1994), 297–301

Loop algebras

Definition

Take $S = \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} S_i$ finite-dim Lie algebra, $U \subseteq S_1$.
The (positive part of the twisted) *loop algebra* of S is the Lie subalgebra of $S \otimes \mathbb{F}[t]$ generated by $U \otimes t$.

- Special case: $S = \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} S_i, \quad \dim(S_i) = 1,$
  $D \in \mathrm{Der}(S), \quad DS_i = S_{i+1}.$
- Then the loop algebra $L = \bigoplus_{i=1}^{\infty} L_i$ of $S \oplus FD$ with respect to $U = S_1 \oplus FD$ is an infinite-dim gLamc:
  $$L_1 = S_1 \otimes t + FD \otimes t, \qquad \text{and}$$
  $$L_i = S_{i \pmod N} \otimes t^i \text{ for } i > 1.$$
- Shalev noted that there exist simple Lie algebras S (certain *Block algebras*) satisfying these requirements.

Many gLamc in prime characteristic

- The picture for gLamc of char $p > 0$ actually looks much more complicated than Shalev's result suggests:

Theorem (Caranti-SM-Newman 1997)

- *Over any field F of char $p > 0$ there are $|F|^{\aleph_0}$ isomorphism types of infinite-dim gLamc.*
- *Most of them are not soluble, and are not even periodic.*
- *The soluble ones alone are* $\max\{|F|, \aleph_0\}$.

A. Caranti, SM and M.F. Newman
Graded Lie algebras of maximal class
Trans. Amer. Math. Soc. **349** (1997), 4021–4051

A classification of gLamc

Theorem (Caranti-Newman 2000)

The constructions given in [Caranti-SM-Newman 1997] produce all the infinite-dimensional gLamc, for $p > 2$.

- A similar result holds in char two [Jurman 2005], with an additional type besides Shalev's algebras.

A. Caranti and M.F. Newman
Graded Lie algebras of maximal class. II
J. Algebra **229** (2000), 750–784

G. Jurman
Graded Lie algebras of maximal class. III
J. Algebra **284** (2005), 435–461

A loop algebra of the Witt algebra

$$W(1;1) = \bigoplus_{i=-1}^{p-2} Fe_i, \quad [e_i, e_j] = (j-i)e_{i+j}$$

is the Witt algebra over F of char p, with its $\mathbb{Z}$-grading:

$$e_{-1} \quad e_0 \quad e_1 \quad e_2 \quad \cdots \quad e_{p-3} \quad e_{p-2}$$

Viewing the degrees modulo $p-1$ and changing their sign, we get a $(\mathbb{Z}/(p-1)\mathbb{Z})$-grading, with 1-component

$$U = S_1 = Fe_{-1} + Fe_{p-2}.$$

Now build the loop algebra:

$$x := e_{-1} \otimes t \qquad y := e_{p-2} \otimes t$$

$$e_{p-3} \otimes t^2$$

$$e_{p-4} \otimes t^3$$

$$\vdots$$

Thin groups and Lie algebras

Definition (Brandl-Caranti-Scoppola 1992)

A *thin* p-group is a p-group $G = \langle x, y \rangle$ such that

$$1 < N \triangleleft G \quad \Rightarrow \quad \gamma_{i+1}(G) \leq N \leq \gamma_i(G) \text{ for some } i.$$

- thin $\Leftrightarrow$ of *width* two and *obliquity* zero
- G is thin $\Leftrightarrow$ the gLa associated to its l.c.s. is thin

Definition

A *thin* Lie algebra is a gLa $L = \bigoplus_{i \geq 1} L_i$, with $\dim L_1 = 2$, which satisfies the *covering property*

$$L_{i+1} = [u, L_1] \text{ for all } 0 \neq u \in L_i, \text{ for all } i \geq 1.$$

- $\dim(L_i) \leq 2$ for all i. If $= 2$, call L_i a *diamond*.
- If L_1 is the only diamond then L is a gLamc. Exclude.
- If $\dim L = \infty$, then L is just-infinite, hence $Z(L) = 0$.

The second diamond in thin groups

Theorem (Caranti-SM-Newman-Scoppola 1996)

- *Let G be an infinite thin pro-p group, and let L be the gLa associated to $\{\gamma_i(G)\}$. If L_k is the second diamond, then k can only be 3, 5, or p.*
- *If $k = 3 < p$ and L_4 is not a diamond, then L has one of two possible isomorphism types.*
- *If $k = 5 < p$, then L is uniquely determined.*

- The case $k = p > 2$ occurs for the gLa associated with the lower central series of the Nottingham group.

A. Caranti, SM, M.F. Newman and C.M. Scoppola
Thin groups of prime-power order and thin Lie algebras
Quart. J. Math. Oxford Ser. (2) **47** (1996), 279–296

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q - 1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

Classical thin Lie algebras

Two of the thin Lie algebras L with $k = 3, 5$ are:

x	y	L_1
	$[yx]$	L_2
$[yxx]$	$[yxy]$	L_3
	$\vdots$ $[yxxy]$	$\vdots$

x	y
	$[yx]$
	$[yxx]$
	$[yxxx]$
$[yxxxx]$	$[yxxxy]$
	$[yxxxxy] =: v$
$[vx]$	$[vy]$
	$[vxy]$
	$[vxyx]$
$\vdots$	$[vxyxx]$

They are loop algebras of classical simple algebras of type A_1 and A_2. For example, the former is a loop algebra of

$$\mathfrak{sl}_2 = Fe_{-1} \oplus Fe_0 \oplus Fe_1, \quad [e_i, e_j] = (j - i)e_{i+j}$$

where the degrees are viewed modulo 2.

The second diamond in thin Lie algebras

Theorem

If L_k is the second diamond of a thin Lie algebra of char $p \neq 2$, then k can only be $3, 5, q$ or $2q - 1$, where q is a power of p.

- This includes char zero, but char two is different and much harder (see later section).

A. Caranti and G. Jurman
Quotients of maximal class of thin Lie algebras. The odd characteristic case
Comm. Algebra **27** (1999), 5741–5748

M. Avitabile and G. Jurman
Diamonds in thin Lie algebras
Boll. Unione Mat. Ital. **4** (2001), no. 3, 597–608

Nonclassical thin Lie algebras

Besides the *classical* cases $k = 3, 5$, in char $p > 0$ we have two more large classes:

- When $k = q$ we have the *Nottingham* algebras. Several have been constructed as loop algebras of
  - Zassenhaus algebras $W(1; n)$, of dim p^n,
  - graded simple Hamiltonian alg. $H(2; \mathbf{n})^{(2)}$, of dim $p^n - 2$,
  - Albert-Zassenhaus algebras $H(2; \mathbf{n}; \Phi(1))$, of dim p^n.
- When $k = 2q - 1$ we have the (-1)*-algebras*. Several have been constructed as loop algebras of
  - Block algebras $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$, of dim $p^n - 1$.

However, both classes contain also (uncountably many) non-periodic algebras, which can be related to gLamc.

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q - 1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

Finite presentations and explicit constructions

Typical results so far have come in pairs:

- A proof that L is uniquely determined (as a thin algebra) by a suitable finite-dim quotient.
  - Done by exhibiting a finite presentation for a central extension $\tilde{L}$ of L.
  - Methods: the Jacobi identity (many times), exploiting the periodicity of the structure.
- An explicit construction of $\tilde{L}$ as a loop algebra of a suitable finite-dimensional Lie algebra S.
  - The centre of $\tilde{L}$ is related to central extensions of S.
  - Methods: guess the right algebra S (easy), and produce a suitable cyclic grading of it (harder).

Diamond types of (-1)-algebras

- Let L be thin with second diamond L_{2q-1}, and $\dim(L) = \infty$.
- Choose generators $x, y \in L_1$, with $[L_2, y] = 0$.
- Let L_h be a diamond.
  Then $L_{h-1} = Fv$ and L_{h+1} are not diamonds.
- We have $[vxy] + [vyx] = 0$ and $[vyy] = 0$, and also

  $$[vyx] = \lambda[vxx] \quad \text{for some } \lambda \in \mathbb{F} \cup \{\infty\}.$$

  We say that L_h is a diamond of type λ.
- When $\lambda = 0$ the element $[vy]$ would be central, hence $[vy] = 0$ and $L_k = F[vx]$ is a *fake* diamond.

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q-1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

All diamonds of infinite type

Theorem (Caranti-SM 1999)

Let L be a (-1)-algebra with all diamonds of type ∞. Then L has a derivation D such that $M = L + FD$ is a graded Lie algebra of maximal class (with respect to a new grading).

- One obtains a bijection of the set of (isom. types of) (-1)-algebras with all diamonds of type ∞ with an explicitly known subset of the gLamc.
- That subset contains $|F|^{\aleph_0}$ isomorphism types. Most of them are non-periodic (hence not loop algebras).

A. Caranti and SM
Some thin Lie algebras related to Albert-Frank algebras and algebras of maximal class
J. Aust. Math. Soc. **67** (1999), 157–184

All diamonds of finite type

Theorem (Caranti-SM 1999)

There is a unique thin Lie algebra L with second diamond L_{2q-1} of finite type. The diamonds occur in all degrees $\equiv 1 \pmod{q-1}$, and their types form an arithmetic progression.

In this description L_q is of type zero, hence *fake*.

Theorem (Caranti-SM 2005)

L is constructed as a loop algebra of a Block algebra $H(2;\mathbf{n};\Phi(\tau))^{(1)}$.

A. Caranti and SM
Gradings of non-graded Hamiltonian algebras
J. Aust. Math. Soc. **79** (2005), 399–440

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q - 1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

Diamonds of both finite and infinite type

Theorem (Avitabile-SM 2005)

- *There is a thin Lie algebra L with (possibly fake) diamonds in all degrees $\equiv 1 \pmod{q-1}$, of type ∞ except those in degree $\equiv q \pmod{p^s(q-1)}$, whose types run over $0, 1, \dots, p-1$ cyclically.*
- *L is a loop algebra of a Block algebra $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$.*
- *L is uniquely determined by a certain finite-dimensional quotient.*

M. Avitabile and SM
Thin Lie algebras with diamonds of finite and infinite type
J. Algebra **293** (2005), 36–64

Hopes for a classification?

Theorem (SM)

Let L be a (-1)-algebra with diamonds of arbitrary types. Then there is a (-1)-algebra $\bar{L}$ with diamonds in the same degrees as L, but all of type ∞.

- Hence the possible degree patterns in which diamonds occur in a (-1)-algebra are known from the theory of gLamc. Going from $\bar{L}$ to L is a deformation problem.
- We expect that as soon as L has at least one diamond of finite type, then it is a loop algebra. Conditional results on the degree of such diamond are obtained in:

M. Avitabile and SM
Diamonds of finite type in thin Lie algebras
J. Lie Theory **19** (2009), 185–207

Diamond types of Nottingham algebras

- For thin algebras with second diamond L_q one can also attach a type $\in F \cup \{\infty\}$ to each diamond, though in a different way.
- Here the type of the second diamond can be set to be -1, and the *fake* diamonds are of two types: 0 and 1.
- In several cases periodicity can be proved, and explicit constructions can be given as loop algebras.
  Then the diamonds occur at regular intervals, and their types follow arithmetic progressions.
- According to the number of fake diamonds found in a period, the finite-dim algebra S of which L is a loop algebra may have dim p^n or $p^n - 2$.

A small zoo

Some diamond type patterns which have been studied are:

- Constant sequence (-1^{∞}); $W(1;n)$, of dim p^n.
- Arithmetic progr. $\subseteq \mathbb{F}_p$; $H(2;\mathbf{n})^{(2)}$, of dim p^n-2.
- Arithmetic progr. $\not\subseteq \mathbb{F}_p$; $H(2;\mathbf{n};\Phi(1))$, of dim p^n.
- $((-1,\infty^{p^r-1})^{\infty})$; $H(2;\mathbf{n};\Phi(1))$, of dim p^n.
- $((*,\infty^{p^r-1})^{\infty})$; $*=$arithmetic progr. (under study)

A. Caranti
J. Algebra **198** (1997), 266–289

M. Avitabile
Internat. J. Algebra Comput. **12** (2002), 535–567

A. Caranti and S. Mattarei
Internat. J. Algebra Comput. **14** (2004), 35–67

M. Avitabile and S. Mattarei
J. Algebra **315** (2007), 824–851

Background and motivations
Graded Lie algebras of finite coclass
Graded Lie algebras of maximal class
Thin Lie algebras
Thinness
Second diamond in degree $2q-1$
Second diamond in degree q
Some special aspects
Characteristic two
Sandwiches

The second diamond in characteristic two

Some of several complications in char two:

- The second diamond of Nottingham algebras in char two has type $-1 = 1$, hence is fake. The second *genuine* diamond L_k may occur later, or not at all!
- The quotient L/L^k is metabelian in odd char (hence $\cong M/M^k$), but not necessarily in characteristic two.

Theorem (Avitabile-SM-Jurman 2010)

- *If L_k is the second* genuine *diamond of a thin Lie algebra of char two, then $k = q - 1$, with q a power of p.*
- *All possibilities for the quotient L/L^k are classified.*

M. Avitabile, SM and G. Jurman
The structure of thin Lie algebras with characteristic two
Internat. J. Algebra Comput **20** (2010), 731–768

Sandwiches

Definition (Kostrikin)

A *sandwich* of a Lie algebra L of char $\neq 2$ is a nonzero element $c \in L$ such that $(\operatorname{ad} c)^2 = 0$.

- The existence of sandwiches distinguishes nonclassical simple finite-dim Lie algebras from the classical ones.
- For example, in the Witt algebra $W(1;1)$,

  $$e_{-1} \quad e_0 \quad e_1 \quad e_2 \quad \cdots \quad e_{p-3} \quad e_{p-2}$$

  any e_i with $i > (p-1)/2$ is a sandwich.
- More generally, any Lie algebra *of Cartan type* $\mathcal{L}$ has a *standard* maximal subalgebra $\mathcal{L}_0$, and *standard* filtration

  $$\mathcal{L} = \mathcal{L}_{-r} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_s \supset 0$$

  with $s > r$ (and $r = 1$ or 2).
  Any nonzero element of $\mathcal{L}_i$ with $i > (s+r)/2$ is a sandwich.

Sandwiches in thin Lie algebras

- $\mathcal{L}_0$ equals the centralizer of the *sandwich* subalgebra, the subalgebra spanned by all sandwiches.
- In particular, $\mathcal{L}_0$ is unique, and hence $\mathrm{Aut}(\mathcal{L})$-invariant.

Theorem (SM)

Let L be a thin Lie algebra of odd char, with second diamond in degree > 5. Then $0 \neq y \in C_{L_1}(L_2)$ is a sandwich.

- This tells apart classical and nonclassical thin Lie algebras.
- Given a thin Lie algebra L, thought to be isomorphic with a loop algebra of some nonclassical simple Lie algebra S, how do we find an explicit isomorphism?
- One way is to identify, within L, the standard maximal subalgebra S_0 of S. Finding sandwiches is crucial.