Non-coprime automorphisms of finite groups

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I. Small centralizers of elements or automorphisms

One of main avenues in group theory: small centralizer $C_G(g) \Rightarrow$ structure of G.

Recall: centralizer $C_G(g) = \{x \in G \mid gx = xg\}$.

Brauer–Fowler (1955): if G is finite and $g \in G$ is an involution, then G has a soluble subgroup of index bounded in terms of $|C_G(g)|$.

Using the classification of finite simple groups, Hartley (1992) generalized Brauer–Fowler theorem for any order of $g \in G$:

G has a soluble subgroup of index bounded in terms of $|C_G(g)|$.

These results can be stated in terms of automorphisms $\varphi \in \operatorname{Aut} G$:

structure of G in terms of φ and the fixed-point subgroup $C_G(\varphi)$

(=centralizer in the semidirect product $G \rtimes \operatorname{Aut} G$).

Coprime automorphisms: solubility

There are especially nice results for finite groups

with coprime automorphisms $\varphi \in \mathrm{Aut}\, G$,

that is, when $(|\varphi|, |G|) = 1$.

Thompson (1959): if $|\varphi|$ is a prime and $C_G(\varphi) = 1$, the G is nilpotent.

Rowley (1995) using CFSG: $C_G(A) = 1$ and $(|G|, |A|) = 1 \Rightarrow G$ is soluble.

Fitting height

Recall that the Fitting subgroup $F(G)$ of a finite group G is the largest normal nilpotent subgroup of G.

The Fitting series of G is defined as $F_1(G) = F(G)$,

and by induction $F_{i+1}(G)$ is the inverse image of $F(G/F_i(G))$.

If G is soluble, then the least number h such that $F_h(G) = G$

is called the Fitting height $h(G)$ of G.

Coprime automorphisms: Fitting height

Once G is soluble, next task is bounding the Fitting height.

Let G be a soluble finite group and $A \leqslant \operatorname{Aut} G$ soluble with $(|G|, |A|) = 1$.

Thompson (1964): the Fitting height $h(G)$ is bounded

in terms of $h(C_G(A))$ and the composition length $\alpha(A)$ of A,

namely, $h(G) \leqslant 5^{\alpha(A)} \cdot h(C_G(A))$.

Bounds in Thompson's theorem were improved in numerous papers:

. . . , Kurzweil (1971) linear bound, . . . , culminating in definitive results by

Turull (1984): $h(G) \leqslant 2\alpha(A) + h(C_G(A))$ and

Hartley–Isaacs (1990): there is a subgroup H of index $\leqslant f(|A|, |C_G(A)|)$ with $h(H) \leqslant 2\alpha(A) + 1$.

Coprime automorphisms: Fitting height w.r.t. rank

Let G be a soluble finite group and $A \leqslant \operatorname{Aut} G$ soluble with $(|G|, |A|) = 1$.

By definition, a finite group G has (Prüfer) rank at most r if every subgroup of G can be generated by r elements.

A rank analogue of the Hartley–Isaacs theorem:

Mazurov–Khukhro (2006): there is $H \leqslant G$ with $|G : H|$ bounded

in terms of $|A|$ and the (Prüfer) rank of $C_G(A)$ (instead of $|C_G(A)|$)

and with $h(H) \leqslant 5(4^{\alpha(A)} - 1)/3$.

(So far, there are no improvements of the bounds towards linear ones similar to those in Kurzeil–Turull–Hartley–Isaacs.)

Non-coprime automorphisms: solubility

Rowley (1995) using CFSG:
$C_G(\varphi) = 1 \Rightarrow G$ is soluble (for any $|\varphi|$).

Brauer–Fowler (1955):
if τ is an involutive automorphism of a finite group G,
then G has a soluble subgroup of index bounded in terms of $|C_G(\tau)|$.

Hartley (1992) using CFSG:
for any automorphism $\varphi \in \mathrm{Aut}\, G$ of a finite group G,
G has a soluble subgroup of index bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Non-coprime automorphisms: Fitting height

...Thus we have reduction to soluble groups.

Next task is bounding the Fitting height.

For $A \leqslant \operatorname{Aut} G$, no such bounds in general!
even in the fixed-point-free case.

Bell and Hartley (1990): for **any non-nilpotent** finite group A

there are soluble finite groups G of arbitrarily large Fitting height

admitting $A \leqslant \operatorname{Aut} G$ with $C_G(A) = 1$.

Nilpotent fixed-point-free group of automorphisms

However, still true for nilpotent fixed-point-free $A \leqslant \operatorname{Aut} G$.

Dade (1969): if G is soluble, and $A \leqslant \operatorname{Aut} G$ is nilpotent with $C_G(A) = 1$,

then $h(G)$ is bounded in terms of $\alpha(A)$

(special case of Dade's theorem on Carter subgroups);

the bound is exponential.

Difficult problem of improving Dade's bound to a linear one,

even for cyclic $A = \langle \varphi \rangle$.

(Some special cases tackled by Ercan and Güloğlu.)

So far the best general result is

Jabara (2017): If $C_G(\varphi) = 1$, then $h(G) \leqslant 7\alpha(\langle \varphi \rangle)^2$.

New result

Theorem 1

If a finite group G admits an automorphism φ of order n having $m = |C_G(\varphi)|$ fixed points, then G has a soluble subgroup of (m, n)-bounded index and (m, n)-bounded Fitting height.

The reduction to the soluble case is by Hartley's generalized Brauer–Fowler theorem based on the classification of finite simple groups.

Corollary 1

If a finite group G contains an element g with centralizer of order $m = |C_G(g)|$, then G has a soluble subgroup of m-bounded index and m-bounded Fitting height.

Application to locally finite groups

Recall: Corollary 1

If a finite group G contains an element g with centralizer of order $m = |C_G(g)|$, then G has a soluble subgroup of m-bounded index and m-bounded Fitting height.

Hence we have an affirmative answer to Hartley's problem of 1995 (Problem 13.8(a) in the "Kourovka Notebook" recorded by Belyaev).

Corollary 2

If a locally finite group G has an element g with finite centralizer of order $m = |C_G(g)|$, then G has a locally soluble subgroup of finite m-bounded index which has a normal series of finite m-bounded length with locally nilpotent factors.

Obtained by inverse limit argument applying Corollary 1 to every finite subgroup of G containing g.

Difficulties in dealing with non-coprime automorphism

Let $\varphi \in \operatorname{Aut} G$ and let N be a φ-invariant normal subgroup.

If φ is coprime, then $C_{G/N}(\varphi) = C_G(\varphi)N/N$.

This is not always true if φ is non-coprime.

Still, we have $|C_{G/N}(\varphi)| \leqslant |C_G(\varphi)|$ for any $|\varphi|$.

If φ is coprime, then there are φ-invariant Sylow p-subgroups, and φ-invariant Hall π-subgroups if G is soluble.

(Recall that H is a Hall π-subgroup of G for a set of primes π if $|H|$ is a π-number, while $|G : H|$ is a π'-number.)

Not always true in general for non-coprime φ.

Idea of the proof

Recall: G soluble, $\varphi \in \operatorname{Aut} G$; need a bound for
the Fitting height $h(G)$ in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Induction on $|\varphi|$ and $|C_G(\varphi)|$.

If $|C_{G/N}(\varphi)| < |C_G(\varphi)|$ for a φ-invariant normal subgroup N,
then induction can be applied to G/N.

Key Lemma

If $|C_{G/N}(\varphi)| = |C_G(\varphi)|$, then N has φ-invariant Hall subgroups.

This lemma enables us to use

Busetto–Jabara (2016)

If $G = UV = VW = UW$ is a finite soluble group for three different Hall subgroups U, V, W, then $h(G) \leqslant h(U) + h(V) + h(W) - 2$.

(Jabara's quadratic bound for $h(G)$ when $C_G(\varphi) = 1$ used this result.)

Base of induction

Recall: G soluble, $\varphi \in \mathrm{Aut}\, G$; need a bound for the Fitting height $h(G)$ in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Hartley (1994, unpubl.) and Khukhro (2015)

proved for $|\varphi| = p^m q^n$.

Essential to cover $|\varphi|$ equal to product of $\leqslant$ **two** prime-powers, since the arguments in the induction step need **three** φ-invariant Hall subgroups as in Busetto–Jabara theorem, as we shall now see.

Induction step

Let $|\varphi| = p^m q^n r^s \cdots$ be a product of at least **three** prime-powers.

Assume $|C_{G/N}(\varphi)| = |C_G(\varphi)|$ for a φ-invariant normal subgroup N.
Then by Key Lemma, N has three φ-invariant Hall subgroups $S_{p'}$, $S_{q'}$, $S_{r'}$ such that $N = S_{p'}S_{q'} = S_{p'}S_{r'} = S_{q'}S_{r'}$.

For each, say, $S_{p'}$, the p-part φ_p has centralizer $C_{S_{p'}}(\varphi_p)$ admitting $\varphi_{p'}$ of smaller order with $\left|C_{C_{S_{p'}}(\varphi_p)}(\varphi_{p'})\right| = |C_{S_{p'}}(\varphi)| \leqslant m$, so $C_{S_{p'}}(\varphi_p)$ has bounded Fitting height by induction on $|\varphi|$.

Then $S_{p'}$ admits **coprime** automorphism φ_p and so $S_{p'}$ has bounded Fitting height by the coprime theorem of Thompson (with better bounds by ...–Turull–Hartley–Isaacs).

Same for $S_{q'}$, $S_{r'}$. Hence, by Busetto–Jabara, $h(N) \leqslant f(m,n)$.

Set $N = F_{f(m,n)+1}(G)$. Either $G = N = F_{f(m,n)+1}(G)$, or $|C_{G/N}(\varphi)| < |C_G(\varphi)|$, and then induction can be applied to G/N. □

Open problems on bounding Fitting height

By analogy with the coprime theorem of Hartley and Isaacs (1990):

Conjecture 1

If a finite group G admits an automorphism φ of order n having $m = |C_G(\varphi)|$ fixed points, then G has a soluble subgroup of (m, n)-bounded index whose Fitting height is n-bounded, or even $\alpha(\langle\varphi\rangle)$-bounded.

So far such a result is only known for automorphisms of prime-power order due to Hartley and Turau (1987) (even in the 'strong' version with $\alpha(\langle\varphi\rangle)$-bounded Fitting height).

Open problems in terms of rank

By analogy with the coprime theorem of Mazurov–Khukhro (2006):

Conjecture 2

If a finite soluble group G admits an automorphism φ of order n such that the fixed point subgroup $C_G(\varphi)$ has (Prüfer) rank r, then the Fitting height of G is (n, r)-bounded.

(...Or even there is a normal subgroup N such that the rank of G/N is (n, r)-bounded, while the Fitting height of N is n-bounded.)

Unlike coprime case and unlike results with order restriction on $C_G(\varphi)$, examples show that here one cannot drop the solubility condition on G.

The affirmative answer to even the 'weak' version of Conjecture 2 would provide an affirmative solution to Hartley's problem 13.8(b) in Kourovka Notebook about a locally soluble locally finite group containing an element with Chernikov centralizer. So far even this weak rank version is only known for $|\varphi| = p^k q^l$ due to Hartley (unpublished, 1994).

II. Local–global generation property of commutators

(Joint work with C. Acciarri, R. M. Guralnick, and P. Shumyatsky.)

Local–global generating property for order

Example (folklore)

Suppose that in a group G there are finitely many commutators, say, m. Then $[G, G]$ is finite and has m-bounded order.

(Commutators are elements of the form $[a, b] = a^{-1}b^{-1}ab$.)

Proposition

Suppose that A is a group of automorphisms (or a subgroup) of a group G such that the set of commutators $\{[g, a] \mid g \in G,\ a \in A\}$ is finite of cardinality m. Then $[G, A]$ is finite of order bounded in terms of m.

Here, $[g, a] = g^{-1}g^{a}$ is a commutator in $G \rtimes A$.

Similarly to Example above, the proof is based on Schur's theorem:

If $G/Z(G)$ is finite of order n, then $[G, G]$ is finite of n-bounded order.

Local–global generating property for rank

Recall that a finite group G has (Prüfer) rank at most r if every subgroup of G can be generated by r elements.

Theorem (C. Acciarri, R. Guralnick, EKh, P. Shumyatsky, as reported at this seminar during my visit to ICM-Shenzhen in 2025)

Suppose that G is a finite group admitting a group of **coprime** automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $\{[g,a] \mid g \in G,\ a \in A\}$ can be generated by r elements. Then $[G,A]$ has r-bounded rank.

"Any subset of $\{[g,a] \mid g \in G,\ a \in A\}$ generates an r-generator subgroup" is a rank analogue of restricting the cardinality of $\{[g,a] \mid g \in G,\ a \in A\}$.

So, 'local' condition of r-generation by subsets of $\{[g,a] \mid g \in G,\ a \in A\}$ implies the 'global' $f(r)$-generation of *all subgroups* of $[G,A]$.

New **non-coprime** result

Theorem 2 (C. Acciarri, R. Guralnick, EKh, P. Shumyatsky)

Suppose that A is a group of automorphisms of a finite $\pi(A)$-soluble group G such that any subset of $\{[g,a] \mid g \in G,\ a \in A\}$ generates a subgroup that can be generated by r elements. Then the rank of $[G,A]$ is bounded in terms of r.

Here, $\pi(A)$ is the set of prime divisors of $|A|$;

$\pi(A)$-solubility means that all non-soluble factors in a chief series have orders coprime to $|A|$.

Examples show that the $\pi(A)$-solubility condition on G cannot be dropped for a non-coprime group of automorphisms A.

The proof of Theorem 2 is rather long and technical; it also depends on the classification of finite simple groups.

Corollary 3

Suppose that A is a group of automorphisms of a finite soluble group G such that any subset of $\{[g,a] \mid g \in G,\ a \in A\}$ generates a subgroup that can be generated by r elements. Then the rank of $[G,A]$ is bounded in terms of r.

We also have a corollary about a rank generation property of commutators with elements of π-subgroups in finite π-soluble groups.

Corollary 4

Let H be a π-subgroup of a finite π-soluble group G such that any subset of $\{[g,h] \mid g \in G,\ h \in H\}$ generates a subgroup that can be generated by r elements. Then the rank of $[G,H]$ is bounded in terms of r.

Corollary 4 is not equivalent to Theorem 2, since the group of automorphisms in this theorem does not have to be π-soluble. As reported at this seminar in 2025, earlier we proved this result when H is a Sylow p-subgroup of a finite p-soluble group G. Examples show that the (π-)solubility condition on G cannot be dropped in Corollaries 3, 4.

Thank you!